Monday 11/8/21

Section 16.2-16.3 : Line Integrals

IDEA: $f: \mathbb{R}^n \to \mathbb{R}$ a function, C a curve in $\mathbb{R}^n$. Understand how f "holds up" along the curve.

$$\begin{array}{ccc} & \mathbb{R}^n & \\ {}^{\vec r(t)}\nearrow & & \searrow^{f} \\ [a,b] & \xrightarrow{f(\vec r(t))} & \mathbb{R} \ldots \end{array}$$

Picture:

① Approximate the curve piecewise linearly

② "Unravel" approximation to an interval

③ Approximate buildup w/ rectangles having height f(left endpoint) and width $|$(r(left endpoint) - r(right endpoint)$|$

④ Limit these approximations by refining the segments

Definition: The line integral of f along curve C parameterized by $\vec r(t)$ on $a \le t \le b$ is:

$$\int_C f\,ds = \int_{t=a}^{b} f(\vec r(t))\,|\vec r\,'(t)|\,dt$$

NB: ① "ds" evokes the idea of arc length...

② if $f = 1$, then $s(C) = \int_C 1\,ds = \int_{t=a}^{b} |\vec r\,'(t)|\,dt$ = arc length of C

Example: Compute $\int_C f\,ds$ for $f(x,y) = x^2+y^2-xy$ and C the upper hemisphere of the unit circle with positive orientation.

Sol: $\int_C f\,ds = \int_{t=a}^{b} f(\vec r(t))\,|\vec r\,'(t)|\,dt$ $\quad \vec r(t) = \langle \cos(t), \sin(t)\rangle$ (because unit circle) $\quad r'(t) = \langle -\sin(t), \cos(t)\rangle$

$0 \le t \le \pi$ (upper hemisphere) $\quad |\vec r\,'(t)| = \sqrt{\sin^2(t)+\cos^2(t)} = 1 \quad f(\vec r(t)) = \cos^2(t)+\sin^2(t) - \cos(t)\sin(t) = 1-\cos(t)\sin(t)$

$$\int_{t=0}^{\pi} (1-\cos(t)\sin(t))\cdot 1\,dt = \begin{pmatrix} u = \cos(t) \\ du = -\sin(t)\,dt \end{pmatrix} \to \left[t + \tfrac{1}{2}\cos^2(t)\right]_{t=0}^{\pi} = \left(\pi + \tfrac12(-1)^2\right) - \left(0 + \tfrac12(1)^2\right) = \pi$$

Definition: For a curve C parameterized by $\vec r(t)$ on $[a,b]$ and $f: \mathbb{R}^n \to \mathbb{R}$ and x_k a variable of f, we define

$$\int_C f\,dx_k = \int_{t=a}^{b} f(\vec r(t))\,x_k'(t)\,dt$$

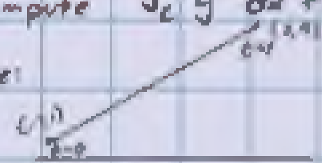
where $x_k'(t)$ is the derivative of the k^{th} term of $\vec r(t)$

$\int_C f\,dx_1 = 0$ because all change is in the x_2 direction.

Example: Compute $\int_C y^2\,dx + \int_C x\,dy$ for C the line segment oriented from $(-7,1)$ to $(5,9)$

Note: a line segment from A to B we can always parameterize as $\vec r(t) = (1-t)A + tB$ for $0 \le t \le 1$

Picture:



Sol: $\vec r(t) = (1-t)\langle -7,1\rangle + t\langle 5,9\rangle = \langle -7+12t, 1+8t\rangle$ on $0 \le t \le 1$ $\quad \vec r\,'(t) = \langle 12, 8\rangle$

$$\int_C y^2\,dx + \int_C x\,dy = \int_{t=0}^{1} (1+8t)^2\,12\,dt + \int_{t=0}^{1} (-7+12t)\cdot 8\,dt = \int_{t=0}^{1} \left(12(1+16t+64t^2) + 8(12t-7)\right)dt$$

$$= 4\int_{t=0}^{1} (3+48t+192t^2+24t-14)\,dt = 4\int_{t=0}^{1} (-11+72t+192t^2)\,dt = 4\left[-11t+36t^2+64t^3\right]_{t=0}^{1} = 4(-11+36+64-0) = 356$$

Definition: The line integral of vector field $\vec v$ along curve C parameterized by $\vec r(t)$ on $[a,b]$ is...

$$\int_C \vec v \cdot d\vec r = \int_{t=a}^{b} \vec v(\vec r(t)) \cdot \vec r\,'(t)\,dt \left(= \int_C \vec v \cdot \vec T\,ds \text{ where } \vec T(t) \text{ is unit tangent vector to } \vec r(t) \text{, i.e. } \vec T(t) = \frac{\vec r\,'(t)}{|\vec r\,'(t)|}\right)$$

Example: Compute $\int_C \vec v \cdot d\vec r$ for $\vec v = \langle xy, yz, xz\rangle$ and C is the curve parameterized by $\vec r(t) = \langle t, t^2, t^3\rangle$ on $0 \le t \le 2$

Sol: $\int_C \vec v \cdot d\vec r = \int_{t=a}^{b} \vec v(\vec r(t)) \cdot \vec r\,'(t)\,dt \qquad r'(t) = \langle 1, 2t, 3t^2\rangle \qquad \vec v(\vec r(t)) = \langle t\,t^2, t^2 t^3, t\,t^3\rangle = \langle t^3, t^5, t^4\rangle$

$$\int_{t=0}^{2} \langle t^3, t^5, t^4\rangle \cdot \langle 1, 2t, 3t^2\rangle = \int_{t=0}^{2} (t^3+2t^6+3t^6)\,dt = \left[\tfrac14 t^4 + \tfrac57 t^7\right]_{t=0}^{2} = \tfrac{16}{4} + \tfrac57\cdot 128 - 0 = \tfrac{668}{7}$$

From Physics: The work done moving a particle along curve C through force field $\vec F$ is $\int_C \vec F \cdot d\vec r$

Example (to do at home): Compute the work done by particle moving along the unit circle counter-clockwise for the quarter-circle through the force field $\vec F = \langle x^2, -xy\rangle$

NB: In example 2: $\int_C y^2\,dx + \int_C x\,dy$ is often abbreviated $\int_C y^2\,dx + x\,dy$. In fact, for any curve C we write $\int_C P\,dx + Q\,dy$ to abbreviate $\int_C P\,dx + \int_C Q\,dy$

Question: Noting that these are 1-variable integrals "twisted up" in $\mathbb{R}^n$, is there an analogue of the FUNDAMENTAL THEOREM OF CALCULUS for use with line integrals?

Bad News: For an integral line $\int_C f\,ds$, there's no reasonable notion of an antiderivative for $f: \mathbb{R}^n \to \mathbb{R}$...

Good News: When $\vec v$ is a conservative vector field, the potential function acts like an antiderivative!

Prop (Fundamental Theorem of Line Integrals [FTLI]): Let $f: \mathbb{R}^n \to \mathbb{R}$ have cts partial derivatives and suppose C is a smooth curve in $\mathbb{R}^n$ parameterized by $\vec r(t)$ on $[a,b]$. Then

$$\int_C \nabla f \cdot d\vec r = f(\vec r(b)) - f(\vec r(a))$$

Proof: We compute $\int_C \nabla f \cdot d\vec r \overset{\text{def of line integral}}{=} \int_{t=a}^{b} \nabla f(\vec r(t)) \cdot \vec r\,'(t)\,dt \overset{\text{multivariable chain rule}}{=} \int_{t=a}^{b} \frac{d}{dt}\left[f(\vec r(t))\right]dt \overset{\text{FTC}}{=} \left[f(\vec r(t))\right]_{t=a}^{b} = f(\vec r(b)) - f(\vec r(a))$

Example: Compute $\int \vec{v} \cdot d\vec{r}$ for $\vec{v} = \langle (1+xy)e^{xy}, x^2e^{xy} \rangle$ on $\vec{r}(t) = \langle \cos(t), 2\sin(t) \rangle$ for $0 \le t \le \pi/2$

Sol: First, verify that $\vec{v}$ is conservative: $\frac{\partial}{\partial y}\left[(1+xy)e^{xy}\right] = (1+xy)e^{xy}x + (x)e^{xy} = e^{xy}(2x + x^2y + 1)$

$\frac{\partial}{\partial x}\left[x^2e^{xy}\right] = 2xe^{xy} + x^2(ye^{xy}) = e^{xy}(2x + x^2y)$ Since the two are equivalent $\vec{v}$ is conservative

$f(x,y) = \int \frac{\partial f}{\partial y}\,dy = \int x^2e^{xy}\,dy = xe^{xy} + C(x)$ $\therefore (1+xy)e^{xy} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[xe^{xy} + C(x)\right] = e^{xy} + xy + C'(x) = (1+xy)e^{xy} + C'(x)$

$\therefore C'(x) = 0$ yields $C(x) = D_{\text{constant}}$ $\therefore$ We have potential function $f(x,y) = xe^{xy} + D$ choose $D = 0$

$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(\pi/2)) - f(\vec{r}(0))$ ⟵ evaluate at home